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EXISTENCE OF PERIODIC SOLUTIONS OF THE GENERALIZED BURGERS SYSTEM

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The boundary value problem for Burgers equations of compressible fluid is considered. Proof is given of the existence of periodic solution as the limit of solutions of initial boundary value problems in which the instant of initial data definition tends to minus infinity.

1. Statement of the problem. The system of equations for a viscous gas is converted into the generalized Burgers system [1] by neglecting in the equations of momentum conservation the pressure gradient. In mass Lagrangian variables [2] that system is of the form

$$u_t = \mu (\rho u_x)_x, \quad v_t = u_x, \quad v = \rho^{-1} \quad (\mu = \text{const} > 0) \tag{1.1}$$

where u is the velocity, ρ is the density, v is the specific volume, and μ is the viscosity coefficient of gas.

Unlike the known Burgers equation and the Burgers turbulence model [3], the generalized system allows for the compressibility of gas. The problems with initial data for Eqs. (1.1) was investigated in [1, 4, 5]. The object of the present investigation is to prove that, if the t-periodic function $f(x, t)$ of period T is present in the right-hand side of the first of Eqs. (1.1), then system (1.1) has a periodic solution of the same period. The existence of periodic solutions for the Burgers model and the Burgers turbulence model was established in [6 − 8].

We introduce the notation

$$Q = \{(x, t): x \in (0, M) = \Omega, \ t \in (-\infty, \infty)\} \quad (M = \text{const} > 0)$$
$$\Gamma = \{(a, t): \ a \in \{0, M\}, \ \ t \in (-\infty, \infty)\}, \ \ \bar{Q} = Q \cup \Gamma$$

Let us consider the problem of determining t-periodic functions $u(x, t)$ and $v(x, t)$ of period T such that satisfy in the band Q Eqs. (1.2) in the classical sense, and that at the boundary of region Q condition (1.3) is fulfilled:

$$u_t = \mu (\rho u_x)_x + f, \quad v_t = u_x, \quad v = \rho^{-1} \tag{1.2}$$

$$u|_\Gamma = 0 \tag{1.3}$$

Function v is furthermore subjected to the constraints

$$v(x, t) > 0, \quad (x, t) \in \bar{Q} \tag{1.4}$$
$$\int_\Omega v(x, t)\, dx = V, \quad t \in (-\infty, \infty) \quad (V = \text{const} > 0)$$

The parameters M and V have the following physical meaning: M is the total mass and V the total volume of gas.

The necessary condition of existence of a periodic solution of system (1.2) is

$$\int_t^{t+T} f(x,s)\,ds = 0, \quad (x,t) \in \bar{Q} \tag{1.5}$$

since it follows from (1.2) that

$$f = \frac{\partial}{\partial t}\left(u - \mu \frac{\partial}{\partial x} \ln v\right) \tag{1.6}$$

In what follows we, therefore, assume that condition (1.5) holds for function f. In solving the problem (1.2) – (1.4), which we shall call the Π - problem, we use the scheme proposed in [9] for linear parabolic equations, i.e. we shall construct the solution of the Π - problem as the limit of initial boundary value problems in which the instant of initial data definition approaches $-\infty$. For this we shall need time uniform estimates of the initial boundary value problems.

2. Estimates of the initial boundary value problem. We shall call the following problem

$$\begin{aligned}
&u_t = \mu(\rho u_x)_x + f, \quad v_t = u_x, \quad (x,t) \in Q_0 \\
&u|_\Gamma = 0; \quad u(x,0) = 0, \quad v(x,0) = VM^{-1} \equiv V_0, \quad x \in \bar{\Omega} \\
&(Q_0 = Q \cap (t > 0), \quad \Gamma_0 = \Gamma \cap (t \geqslant 0))
\end{aligned} \tag{2.1}$$

the K -problem.

If f is a fairly smooth function, there exists on any interval of time a unique smooth solution of the K-problem with a positive function v [4, 5], and, depending on the smoothness of f the smoothness of solution can be any. For example, if $f \in H^{\alpha,\alpha/2}(Q_0)$, then

$$u \in H^{2+\alpha,\, 1+\alpha/2}(Q_0), \; v \in H^{1+\alpha,\, 1+\alpha/2}(Q_0) \quad (0 < \alpha < 1)$$

Subsequently we assume that $f \in H^{\alpha,\alpha/2}(\bar{Q})$.

Lemma 1. Let u, v be a solution of the K-problem. Then the estimate

$$|u|_0 \leqslant (1+V)|f|_0^{1/2} \quad \left(|f|_0 = \sup_{\bar{Q}_0}|f|, \quad \bar{Q}_0 = \bar{Q} \cap (t \geqslant 0)\right) \tag{2.2}$$

is valid for function u.

Proof. We set

$$\omega_i = 1 + (i-1)V + (3-2i)\int_0^x v(y,t)\,dy, \quad z_i = \frac{u}{\omega_i}, \quad i = 1, 2$$

Function z_i is the solution of the problem

$$\begin{aligned}
&[z_{it} - \mu(\rho z_{ix})_x]\omega_i = (2i-3)z_i^2\omega_i + 2\mu\rho z_{ix}\omega_{ix} + f, \quad (x,t) \in Q_0 \\
&z_i|_{\Gamma_0} = 0, \quad z_i(x,0) = 0, \quad i = 1, 2
\end{aligned}$$

Applying to it the maximum principle, taking into account that $1 \leqslant \omega_i \leqslant 1 + V$, we obtain

$$(3-2i)z_i \leqslant |f|_0^{1/2} \quad (i = 1, 2)$$

from which follows estimate (2.2).

In what follows letter c denotes a constant positive quantity that depends only on μ, M, V, T, and f.

Lemma 2. If u, v is a solution of the K-problem, the relation

$$0 < c^{-1} \leqslant v \leqslant c \tag{2.3}$$

holds for function v.

Proof. Integration of (1.6) with respect to t from zero to any $t > 0$ with condition (1.5) and estimate (2.2) yields

$$|(\ln v)_x|_0 \leqslant c \tag{2.4}$$

Moreover, owing to the integral constraint (1.4), there exists an $x_0(t) \in \overline{\Omega}$ such that $v(x_0(t), t) = V_0$. Hence the equality

$$\ln v = \ln V_0 + \int\limits_{x_0(t)}^{x} \frac{\partial}{\partial x} \ln v dx$$

and estimate (2.4) prove Lemma 2, and the same estimate (2.4) then yields

$$|v_x|_0 \leqslant c \tag{2.5}$$

Lemma 3. The solution u, v of the K-problem conforms to the inequalities

$$\sup_{t\geqslant 0} \int\limits_{\Omega} u_x^2(x,t)\, dx \leqslant c, \quad \sup_{t\geqslant 0} \iint\limits_{Q_{t,t+1}} (u_{xx}^2 + v_{xt}^2)\, dx\, dt \leqslant c$$
$$(Q_{t_1,t_2} = \Omega \times (t_1, t_2))$$

Proof. Scalar multiplication of the first of Eqs. (2.1) in $L_2(\Omega)$ by u in conjunction with the previously obtained estimates shows that

$$\iint\limits_{Q_{t,t+1}} u_x^2 dx dt \leqslant c$$

We now carry out scalar multiplication of the first of Eqs. (2.1) in $L_2(\Omega)$ by u_{xx} which yields

$$\frac{1}{2}\frac{d}{dt} I(t) + \mu \int\limits_{\Omega} \rho u_{xx}^2 dx = -\int\limits_{\Omega} (f u_{xx} + \mu \rho_x u_x u_{xx})\, dx, \quad I(t) = \int\limits_{\Omega} u_x^2 dx \tag{2.6}$$

and then, having increased the right-hand side of (2.6) using Young's inequalities and decreased the left-hand side using estimate (2.3), obtain

$$\frac{d}{dt} I(t) + \int\limits_{\Omega} u_{xx}^2 dx \leqslant c \int\limits_{\Omega} (u_x^2 + f^2)\, dx \tag{2.7}$$

It follows from this that $I(t) \leqslant c$ for $t \in [0, 1]$. Let us prove that this relation is also valid for $t > 1$. We fix any arbitrary $t_1 > 1$ and introduce a reasonably smooth function $\eta(t)$ with the following properties: $0 \leqslant \eta \leqslant 1$, $\eta = 0$ for $t \leqslant t_1 - 1$, $\eta = 1$ for $t \geqslant t_1$, $|\eta'(t)| \leqslant 2$. We multiply (2.7) by η and integrate from $t_1 - 1$ to t_1, and obtain $I(t_1) \leqslant c$. The first statement of the lemma is proved by the arbitrariness of $t_1 \in (1, \infty)$, while the second is an obvious corollary of (2.7) and (2.1).

Theorem 1. Let u, v be the solution of the K-problem. Then there exists a $\beta \in (0, 1)$ such that the estimate

$$0 < c^{-1} \leqslant v \leqslant c, \quad |v|^{(1+\beta)}_{\bar{Q}_0} \leqslant c, \quad |u|^{(\beta)}_{\bar{Q}_0} \leqslant c \qquad (2.8)$$
$$\sup_{t\geqslant 0} |v|^{(1+\alpha)}_{\bar{Q}_{t,t+1}} \leqslant c, \quad \sup_{t\geqslant 0} |u|^{(2+\alpha)}_{\bar{Q}_{t,t+1}} \leqslant c$$

is valid.

Proof. The first assertion in (2.8) was proved in Lemma 2. Moreover, since

$$\sup_{t\geqslant 0} \sup_{x_1, x_2 \in \bar{\Omega}} |u(x_1, t) - u(x_2, t)| \, |x_1 - x_2|^{-1/2} \leqslant \sup_{t\geqslant 0} \left(\int_{\Omega} u_x^2 dx \right)^{1/2}$$

$$\sup_{x\in\bar{\Omega}} \sup_{t_1, t_2 \geqslant 0} |v(x, t_1) - v(x, t_2)| \, |t_1 - t_2|^{-1/2} \leqslant$$
$$c \sup_{t\geqslant 0} \int\int_{Q_{t,t+1}} (v_t^2 + v_{xt}^2)\, dxdt + 2|v|_0$$

hence, taking into account (2.2), (1.6), and (2.5) and Lemma 3.1 from [10], we obtain the second and third inequalities of (2.8).

To prove the last two estimates in (2.8) let us consider the first of Eqs. (2.1) as a linear parabolic equation for u. It was shown in [10] that

$$|u|^{(2+\gamma)}_{\bar{Q}_{0,2}} \leqslant c|f|^{(\gamma)}_{\bar{Q}_{0,2}}, \quad \gamma = \min(\alpha, \beta) \qquad (2.9)$$

We multiply the first of Eqs. (2.1) by the introduced earlier function η and, denoting $z = \eta u$, as in (2.9) we have

$$|z|^{(2+\gamma)}_{Q_{t_1-1,\, t_1+1}} \leqslant c\left(|f\eta|^{(\gamma)}_{\bar{Q}_{t_1-1,\, t_1+1}} + |\eta' u|^{(\gamma)}_{\bar{Q}_{t_1-1,\, t_1+1}}\right), \quad t_1 \geqslant 1 \qquad (2.10)$$

But since $z = u$ when $t \geqslant t_1$ and

$$|z|^{(2+\gamma)}_{\bar{Q}_{t_1,\, t_1+1}} \leqslant |z|^{(2+\gamma)}_{\bar{Q}_{t_1-1,\, t_1+1}}$$

it follows from (2.10) that

$$|u|^{(2+\gamma)}_{\bar{Q}_{t_1,\, t_1+1}} \leqslant c$$

which together with (2.9) yields

$$|u|^{(2+\gamma)}_{\bar{Q}_{t,\, t+1}} \leqslant c, \quad t \geqslant 0 \qquad (2.11)$$

After this the fourth estimate in (2.8) can be considered as proved by reverting to formula (1.6), and the proof of the fifth estimate is similar to that of estimate (2.11).

Corollary. Obviously

$$|u|_{Q_{t,t+h}}^{(2+\alpha)} \leqslant c(h), \quad |v|_{Q_{t,t+h}}^{(1+\alpha)} \leqslant c(h), \quad t \geqslant 0, \ h > 0$$

since c is independent of t.

3. Existence of a periodic solution. Let $\{t_n\}$ be a sequence such that $t_n \to -\infty$ $(n \to \infty)$ and $\{Q_n\}$ be a sequence of cylinders. We define u^n, v^n as the classic solution in Q_n of the problem

$$\begin{gathered} u_t^n = \mu (\rho^n u_x^n)_x + f, \quad v_t^n = u_x^n \quad (Q_n = \Omega \times (t_n, \infty)) \\ u^n(x, t_n) = 0, \ v^n(x, t_n) = V_0, \quad u^n|_{\Gamma_n} = 0 \quad (\Gamma_n = \Gamma \cap (t \geqslant t_n)) \end{gathered} \tag{3.1}$$

Let us consider sequencies $\{u^n\}$ and $\{v^n\}$ on a fixed compact $\bar{Q}^k = \bar{\Omega} \times [-k, k]$. Since for $t_n < -k$

$$|u^n|_{\bar{Q}^k}^{(2+\alpha)} \leqslant c(k), \quad |v^n|_{\bar{Q}^k}^{(1+\alpha)} \leqslant c(k)$$

hence using the compactness of imbedding

$$H^{i+\alpha, (i+\alpha)/2}(\bar{Q}^k) \to H^{i+\nu, (i+\nu)/2}(\bar{Q}^k) \ (i = 1, 2)$$

for $0 < \nu < \min\{\alpha, \beta\}$, we construct subsequencies $\{u_k^n\}$ and $\{v_k^n\}$ such that for some functions u_k° and v_k°

$$|u_k^n - u_k^\circ|_{\bar{Q}^k}^{(2+\nu)} \to 0, \ |v_k^n - v_k^\circ|_{\bar{Q}^k}^{(1+\nu)} \to 0 \ (n \to \infty) \tag{3.2}$$

with the following valid estimates:

$$\begin{gathered} |u_k^\circ|_{\bar{Q}^k}^{(\nu)} \leqslant c_0, \ |v_k^\circ|_{\bar{Q}^k}^{(1+\nu)} \leqslant c_0, \ 0 < c_0^{-1} \leqslant v_k^\circ \leqslant c_0 \\ u_k^\circ \in H^{2+\nu, 1+\nu/2}(\bar{Q}^k), \quad v_k^\circ \in H^{1+\nu, (1+\nu)/2}(\bar{Q}^k) \end{gathered} \tag{3.3}$$

with the majorant c_0 independent of k. Let us consider sequencies $\{u_k^n\}$ and $\{v_k^n\}$ on compact $\bar{Q}^{k+1}$. As previously, it is again possible to separate the subsequencies $\{u_{k+1}^n\} \subset \{u_k^n\}$ and $\{v_{k+1}^n\} \subset \{v_k^n\}$ such that after the substitution of $k+1$ for k, relations (3.2) and estimates (3.3) are valid. Obviously the derived pairs $\{u_k^\circ, v_k^\circ\}$ and $\{u_{k+1}^\circ, v_{k+1}^\circ\}$ are the same in $\bar{Q}^k$.

Proceeding in the smae manner we construct for any component $\bar{Q}^k$ sequencies $\{u_k^n\}$ and $\{v_k^n\}$ and functions u_k° and v_k° that possess properties (3.2), (3.3), and

$$u_k^\circ = u_{k+1}^\circ, \ v_k^\circ = v_{k+1}^\circ, \quad (x, t) \in \bar{Q}^k \tag{3.4}$$

with u_k^n, v_k^n being the solution of problem (3.1) for some t_n dependent on k.

We define functions u° and v° as follows:

$$u^\circ(x, t) = u_k^\circ(x, t), \quad v^\circ(x, t) = v_k^\circ(x, t), \ (x, t) \in \bar{Q}^k$$

Thereby u° and v° have been determined over the whole set $\bar{Q}$ and, owing to (3.4), their above definition is correct. We shall show that u°, v° is a solution of the Π-problem. It follows from (3.2) that functions u° and v° satisfy Eqs. (1.2) on any bounded set Q^k, hence u°, v° is the classic solution of system (1.2) over the whole set Q. Conditions (1.3) and (1.4) are obviously satisfied. We have moreover the estimates

$$|u^\circ|_{\bar{Q}}^{(\nu)} \leqslant c_0, \quad |v^\circ|_{\bar{Q}}^{(1+\nu)} \leqslant c_0, \quad 0 < c_0^{-1} \leqslant v^\circ \leqslant c_0 \tag{3.5}$$

Let us prove that u° and v° are T-periodic functions. Since function $u_1(x, t) = u^\circ(x, t + T)$ and $v_1(x, t) = v^\circ(x, t + T)$ also satisfy formulas (1.2) – (1.4), hence owing to (1.5), we have

$$u = \mu\rho^\circ v_x - \mu\rho^\circ\rho_1 v_{1x} v, \quad v_t = u_x \quad (u = u^\circ - u_1, v = v^\circ - v_1) \tag{3.6}$$

from which

$$w_t = \mu\rho^\circ w_{xx} - \mu\rho^\circ\rho_1 v_{1x} w_x, \quad w|_\Gamma = 0 \left(w = \int_0^x v\, dx\right) \tag{3.7}$$

Equation (3.7) may be considered to be a linear parabolic equation in w with coefficients whose Hölder norms for the entire band $\bar{Q}$ are bounded. It was shown in [9] that its solution with uniform boundary conditions and bounded throughout the band $\bar{Q}$ can only be zero. Hence $w \equiv 0$ and consequently also $v \equiv 0$. Taking this into consideration we obtain from (3.6) that $u \equiv 0$, and thus prove that u°, v° is a solution of the Π-problem.

The periodicity of functions u° and v° enables us to reinforce estimates (3.5). Let $Q_T = \Omega \times (0, T)$, then from (3.2) we have

$$|u^\circ|_{\bar{Q}_T}^{(2+\nu)} \leqslant c, \quad |v^\circ|_{\bar{Q}_T}^{(1+\nu)} \leqslant c.$$

Repeating the reasoning used in proving Theorem 1 we obtain

$$|u^\circ|_{\bar{Q}_T}^{(2+\alpha)} \leqslant c, \quad |v^\circ|_{\bar{Q}_T}^{(1+\alpha)} \leqslant c, \quad 0 < c^{-1} \leqslant v^\circ \leqslant c \tag{3.8}$$

Let us formulat the final result.

Theorem 2. Let the t-periodic function $f \in H^{\alpha,\alpha/2}(\bar{Q}_T)$ of period T satisfy condition (1.5). There exists then a solution of problem (1.2) – (1.4) which is t-periodic (of period T) for which the estimates (3.8) are valid.

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